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CMB with Quintessence: Analytic Approach and CMBFAST

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Abstract

A particular kind of quintessence is considered, with equation of motion $p_Q/\rho_Q = -1$, corresponding to a cosmological term with time-dependence $\Lambda(t) = \Lambda(t_0)(R(t_0)/R(t))^P$ which we examine initially for $0 \leq P < 3$. Energy conservation is imposed, as is consistency with big-bang nucleosynthesis, and the range of allowed P is thereby much restricted to $0 \leq P < 0.2$. The position of the first Doppler peak is computed analytically and the result combined with analysis of high-Z supernovae to find how values of Ω_m and Ω_Λ depend on P. Some comparison is made to the CMBFAST public code.

Our knowledge of the universe has changed dramatically even in the last five years. Five years ago the best guess, inspired partially by inflation, for the makeup of the present cosmological energy density was $\Omega_m = 1$ and $\Omega_\Lambda = 0$. However, the recent experimental data on the cosmic background radiation and the high - Z (Z = red shift) supernovae strongly suggest that both guesses were wrong. Firstly $\Omega_m \simeq 0.3 \pm 0.1$. Second, and more surprisingly, $\Omega_\Lambda \simeq 0.7 \pm 0.2$. The value of Ω_Λ is especially unexpected for two reasons: it is non-zero and it is ≥ 120 orders of magnitude below its "natural" value.

The fact that the present values of Ω_m and Ω_Λ are of comparable order of magnitude is a "cosmic coincidence" if Λ in the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} + \Lambda g_{\mu\nu}$$

is constant. Extrapolate the present values of Ω_m and Ω_Λ back, say, to redshift $Z = 100$. Suppose for simplicity that the universe is flat $\Omega_C = 0$ and that the present cosmic parameter values are $\Omega_m = 0.300...$ exactly and $\Omega_\Lambda = 0.700...$ exactly. Then since $\rho_m \propto R(t)^{-3}$ (we can safely neglect radiation), we find that $\Omega_m \simeq 0.9999..$ and $\Omega_\Lambda \simeq 0.0000..$ at $Z = 100$. At earlier times the ratio Ω_Λ/Ω_m becomes infinitesimal. There is nothing to exclude these values but it does introduce a second "flatness" problem because, although we can argue for $\Omega_m + \Omega_\Lambda = 1$ from inflation, the comparability of the present values of Ω_m and Ω_Λ cries out for explanation.

In the present paper we shall consider a specific model of quintessence. In its context we shall investigate the position of the first Doppler peak in the Cosmic Microwave Background (CMB) analysis using results published by two of us with Rohm earlier [1]. Other works on the study of CMB include [2–5]. We shall explain some subtleties of the derivation given in [1] that have been raised since its publication mainly because the formula works far better than its expected order-of-magnitude accuracy. Data on the CMB have been provided recently in [6–13] and especially in [14].

The combination of the information about the first Doppler peak and the complementary analysis of the deceleration parameter derived from observations of the high-red-shift supernovae [15,16] leads to fairly precise values for the cosmic parameters Ω_m and Ω_Λ. We shall therefore also investigate the effect of quintessence on the values of these parameters.

In [1], by studying the geodesics in the post-recombination period a formula was arrived at for the position of the first Doppler peak, l_1. For example, in the case of a flat universe with $\Omega_C = 0$ and $\Omega_M + \Omega_\Lambda = 1$ and for a conventional cosmological constant:

$$l_1 = \pi \left(\frac{R_t}{R_0}\right) \left[\Omega_M \left(\frac{R_0}{R_t}\right)^3 + \Omega_\Lambda\right]^{1/2} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}} \tag{1}$$

If $\Omega_C < 0$ the formula becomes

$$l_1 = \frac{\pi}{\sqrt{-\Omega_C}} \left(\frac{R_t}{R_0}\right) \left[\Omega_M \left(\frac{R_0}{R_t}\right)^3 + \Omega_\Lambda + \Omega_C \left(\frac{R_0}{R_t}\right)^2\right]^{1/2} \sin\left(\sqrt{-\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}}\right) \tag{2}$$

For the third possibility of a closed universe with $\Omega_C > 0$ the formula is:

$$l_1 = \frac{\pi}{\sqrt{\Omega_C}} \left(\frac{R_t}{R_0}\right) \left[\Omega_M \left(\frac{R_0}{R_t}\right)^3 + \Omega_\Lambda + 5\Omega_C \left(\frac{R_0}{R_t}\right)^2\right]^{1/2} \sinh\left(\sqrt{\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}}\right) \tag{3}$$

The use of these formulas gives iso-l_1 lines on a $\Omega_M - \Omega_\Lambda$ plot in $25 \sim 50\%$ agreement with the corresponding results found from computer code. On the insensitivity of l_1 to other variables, see [17,18]. The derivation of these formulas was given in [1]. Here we add some more details.

The formula for l_1 was derived from the relation $l_1 = \pi/\Delta\theta$ where $\Delta\theta$ is the angle subtended by the horizon at the end of the recombination transition. Let us consider the Legendre integral transform which has as integrand a product of two factors, one is the temperature autocorrelation function of the cosmic background radiation and the other factor is a Legendre polynomial of degree l. The issue is what is the lowest integer l for which the two factors reinforce to create the doppler peak? For small l there is no reinforcement because the horizon at recombination subtends a small angle about one degree and the CBR fluctuations average to zero in the integral of the Legendre transform. At large l the Legendre polynomial itself fluctuates with almost equispaced nodes and antinodes. The node-antinode spacing over which the Legendre polynomial varies from zero to a local maximum in magnitude is, in terms of angle, on average π divided by l. When this angle coincides with the angle subtended by the last-scattering horizon, the fluctuations of the two integrand factors are, for the first time with increasing l, synchronized and reinforce (constructive interference) and the corresponding partial wave coefficient is larger than for slightly smaller or slightly larger l. This explains the occurrence of π in the equation for the l_1 value of the first doppler peak written as $l_1 = \pi/\Delta\theta$.

Another detail concerns whether to use the photon or acoustic horizon, where the former is $\sqrt{3}$ larger than the latter? If we examine the evolution of the recombination transition given in [19] the degree of ionization is 99% at $5,000^0$K (redshift $Z = 1,850$) falling to 1% at $3,000^0$K ($Z = 1,100$). One can see qualitatively that during the recombination transition the fluctuation can grow. The agreement of the formula for l_1, using the photon horizon, with experiment shows phenomenologcally that the fluctuation does grow during the recombination transition and that is why there is no full factor of $\sqrt{3}$, as would arise using the acoustic horizon, in its numerator. When we look at the CMBFAST code below, we shall find a factor in l_1 of ~ 1.22, intermediate between 1 (optical) and $\sqrt{3}$ (acoustic).

To introduce our quintessence model as a time-dependent cosmological term, we start from the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda(t)g_{\mu\nu} + 8\pi G T_{\mu\nu} = 8\pi G \mathcal{T}_{\mu\nu} \tag{4}$$

where $\Lambda(t)$ depends on time as will be specified later and $T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$. Using the Robertson-Walker metric, the '00' component of Eq.(4) is

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G\rho}{3} + \frac{1}{3}\Lambda \tag{5}$$

while the 'ii' component is

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi Gp + \Lambda \tag{6}$$

Energy-momentum conservation follows from Eqs.(5,6) because of the Bianchi identity $D^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}) = D^\mu(\Lambda g_{\mu\nu} + 8\pi GT_{\mu\nu}) = D^\mu \mathcal{T}_{\mu\nu} = 0$.

Note that the separation of $\mathcal{T}_{\mu\nu}$ into two terms, one involving $\Lambda(t)$, as in Eq(4), is not meaningful except in a phenomenological sense because of energy conservation.

In the present cosmic era, denoted by the subscript '0', Eqs.(5,6) become respectively:

$$\frac{8\pi G}{3}\rho_0 = H_0^2 + \frac{k}{R_0^2} - \frac{1}{3}\Lambda_0 \tag{7}$$

$$-8\pi Gp_0 = -2q_0H_0^2 + H_0^2 + \frac{k}{R_0^2} - \Lambda_0 \tag{8}$$

where we have used $q_0 = -\frac{\ddot{R}_0}{R_0H_0^2}$ and $H_0 = \frac{\dot{R}_0}{R_0}$.

For the present era, $p_0 \ll \rho_0$ for cold matter and then Eq.(8) becomes:

$$q_0 = \frac{1}{2}\Omega_M - \Omega_\Lambda \tag{9}$$

where $\Omega_M = \frac{8\pi G\rho_0}{3H_0^2}$ and $\Omega_\Lambda = \frac{\Lambda_0}{3H_0^2}$.

Now we can introduce the form of $\Lambda(t)$ we shall assume by writing

$$\Lambda(t) = bR(t)^{-P} \tag{10}$$

where b **is** a constant and the exponent P we shall study for the range $0 \leq P < 3$. This motivates the introduction of the new variables

$$\tilde{\Omega}_M = \Omega_M - \frac{P}{3-P}\Omega_\Lambda, \qquad \tilde{\Omega}_\Lambda = \frac{3}{3-P}\Omega_\Lambda \tag{11}$$

It is unnecessary to redefine Ω_C because $\tilde{\Omega}_M + \tilde{\Omega}_\Lambda = \Omega_M + \Omega_\Lambda$. The case $P = 2$ was proposed, at least for late cosmological epochs, in [20].

The equations for the first Doppler peak incorporating the possibility of non-zero P are found to be the following modifications of Eqs.(1,2,3). For $\Omega_C = 0$

$$l_1 = \pi \left(\frac{R_t}{R_0}\right) \left[\tilde{\Omega}_M \left(\frac{R_0}{R_t}\right)^3 + \tilde{\Omega}_\Lambda \left(\frac{R_0}{R_t}\right)^P\right]^{1/2} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P}} \tag{12}$$

If $\Omega_C < 0$ the formula becomes

$$\begin{aligned} l_1 = & \frac{\pi}{\sqrt{-\Omega_C}} \left(\frac{R_t}{R_0}\right) \left[\tilde{\Omega}_M \left(\frac{R_0}{R_t}\right)^3 + \tilde{\Omega}_\Lambda \left(\frac{R_0}{R_t}\right)^P + \Omega_C \left(\frac{R_0}{R_t}\right)^2\right]^{1/2} \times \\ & \times \sin\left(\sqrt{-\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P + \Omega_C w^2}}\right) \end{aligned} \tag{13}$$

For the third possibility of a closed universe with $\Omega_C > 0$ the formula is:

$$\begin{aligned} l_1 = & \frac{\pi}{\sqrt{\Omega_C}} \left(\frac{R_t}{R_0}\right) \left[\tilde{\Omega}_M \left(\frac{R_0}{R_t}\right)^3 + \tilde{\Omega}_\Lambda \left(\frac{R_0}{R_t}\right)^P + \Omega_C \left(\frac{R_0}{R_t}\right)^2\right]^{1/2} \times \\ & \times \quad \sinh\left(\sqrt{\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P + \Omega_C w^2}}\right) \end{aligned} \tag{14}$$

The dependence of l_1 on P is illustrated for constant $\Omega_M = 0.3$ in Fig. 1(a), and for the flat case $\Omega_C = 0$ in Fig. 1(b). For illustration we have varied $0 \leq P < 3$ but as will become clear later in the paper (see Fig 3 below) only the much more restricted range $0 \leq P < 0.2$ is possible for a fully consistent cosmology when one considers evolution since the nucleosynthesis era.

We have introduced P as a parameter which is real and with $0 \leq P < 3$. For $P \to 0$ we regain the standard cosmological model. But now we must investigate other restrictions already necessary for P before precision cosmological measurements restrict its range even further.

Only for certain P is it possible to extrapolate the cosmology consistently for all $0 < w = (R_0/R) < \infty$. For example, in the flat case $\Omega_C = 0$ which our universe seems to approximate [14], the formula for the expansion rate is

$$\frac{1}{H_0^2}\left(\frac{\dot{R}}{R}\right)^2 = \tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P \tag{15}$$

This is consistent as a cosmology only if the right-hand side has no zero for a real positive $w = \hat{w}$. The root $\hat{w}$ is

$$\hat{w} = \left(\frac{3(1-\Omega_M)}{P - 3\Omega_M}\right)^{\frac{1}{3-P}} \tag{16}$$

If $0 < \Omega_M < 1$, consistency requires that $P < 3\Omega_M$.

In the more general case of $\Omega_C \neq 0$ the allowed regions of the $\Omega_M - \Omega_\Lambda$ plot for $P = 0, 1, 2$ are displayed in Fig. 2.

We see from Eq.(16) that if we do violate $P < 3\Omega_M$ for the flat case then there is a $\hat{w} > 0$ where the cosmology undergoes a bounce, with $\dot{R} = 0$ and $\dot{R}$ changing sign. This necessarily arises because of the imposition of $D^\mu \mathcal{T}_{\mu\nu} = 0$ for energy conservation. For this example it occurs in the past for $\hat{w} > 1$. The consistency of big bang cosmology back to the time of nucleosynthesis implies that our universe has not bounced for any $1 < \hat{w} < 10^9$. It is also possible to construct cosmologies where the bounce occurs in the future! Rewriting Eq.(16) in terms of Ω_Λ:

$$\hat{w} = \left(\frac{3\Omega_\Lambda}{3\Omega_\Lambda - (3-P)}\right)^{\frac{1}{3-P}} \tag{17}$$

If $P < 3$, then any $\Omega_\Lambda < 0$ will lead to a solution with $0 < \hat{w} < 1$ corresponding to a bounce in the future. If $P > 3$ the condition for a future bounce is $\Omega_\Lambda < -\left(\frac{P-3}{3}\right)$. What this means is that for the flat case $\Omega_C = 0$ with quintessence $P > 0$ it is possible for the future cosmology to be qualitatively similar to a non-quintessence closed universe where $\dot{R} = 0$ at a finite future time with a subsequent big crunch.

Another constraint on the cosmological model is provided by nucleosynthesis which requires that the rate of expansion for very large w does not differ too much from that of the standard model.

The expansion rate for $P = 0$ coincides for large w with that of the standard model so it is sufficient to study the ratio:

$$(\dot{R}/R)^2_P/(\dot{R}/R)^2_{P=0} \stackrel{w \to \infty}{\to} (3\Omega_M - P)/((3-P)\Omega_M) \tag{18}$$

$$\stackrel{w \to \infty}{\to} (4\Omega_R - P)/((4-P)\Omega_R) \tag{19}$$

where the first limit is for matter-domination and the second is for radiation-domination (the subscript R refers to radiation).

The overall change in the expansion rate at the BBN era is therefore

$$(\dot{R}/R)^2_P/(\dot{R}/R)^2_{P=0} \stackrel{w \to \infty}{\to} (3\Omega_M - P)/((3-P)\Omega_M) \times (4\Omega_R^{trans} - P)/((4-P)\Omega_R^{trans}) \tag{20}$$

where the superscript "trans" refers to the transition from radiation domination to matter domination. Putting in the values $\Omega_M = 0.3$ and $\Omega_R^{trans} = 0.5$ leads to $P < 0.2$ in order that the acceleration rate at BBN be within 15% of its value in the standard model, equivalent to the contribution to the expansion rate at BBN of one chiral neutrino flavor.

Thus the constraints of avoiding a bounce ($\dot{R} = 0$) in the past, and then requiring consistency with BBN leads to $0 < P < 0.2$.

We may now ask how this restricted range of P can effect the extraction of cosmic parameters from observations. This demands an accuracy which has fortunately begun to be attained with the Boomerang data [14]. If we choose $l_1 = 197$ and vary P as $P = 0, 0.05, 0.10, 0.15, 0.20$ we find in the enlarged view of Fig 3 that the variation in the parameters Ω_M and Ω_Λ can be as large as $\pm 3\%$. To guide the eye we have added the line for deceleration parameter $q_0 = -0.5$ as suggested by [15,16]. In the next decade, inspired by the success of Boomerang (the first paper of true precision cosmology) surely the sum $(\Omega_M + \Omega_\Lambda)$ will be examined at much better than $\pm 1\%$ accuracy, and so variation of the exponent of P will provide a useful parametrization of the quintessence alternative to the standard cosmological model with constant Λ.

Clearly, from the point of view of inflationary cosmology, the precise vanishing of $\Omega_C = 0$ is a crucial test and its confirmation will be facilitated by comparison models such as the present one.

We have also studied the use of the public code CMBFAST [21] and how its normalization compares to that in [1]. For example, with P = 0 and $\Omega_\Lambda = 0.3, h_{100} = 0.65$ we find using CMBFAST that

$\Omega_\Lambda = 0.5, l_1 = 284$ ($l_1 = 233$ from [1])

$\Omega_\Lambda = 0.6, l_1 = 254$ ($l_1 = 208$ from [1])

$\Omega_\Lambda = 0.7, l_1 = 222$ ($l_1 = 182$ from [1])

$\Omega_\Lambda = 0.8, l_1 = 191$ ($l_1 = 155$ from [1])

The CMBFAST l_1 values are consistently ~ 1.22 times the l_1 values from [1]. As mentioned earlier, this normalization is intermediate between that for the acoustic horizon ($\sqrt{3}$) and the photon horizon (1).

Finally, we remark that the quintessence model considered here is in the right direction to ameliorate the "age problem" of the universe. Taking the age as 14.5Gy for $\Omega_M = 0.3$, $\Omega_C = 0$ and $h_{100} = 0.65$ the age increases monotonically with P. It reaches slightly over 15 Gy at the highest-allowed value $P = 0.2$. This behavior is illustrated in Fig 4 which assumes $\Omega_M = 0.3$ and flatness as P is varied.

Acknowledgments

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Figure 1.

Dependence of l_1 on P for (a) fixed $\Omega_M = 0.3$; (b) fixed $\Omega_C = 0$.

Figure 2.

Regions of the $\Omega_M - \Omega_\Lambda$ plot where there is a future bounce (small dot lattice), no bounce (unshaded) and a past bounce (large dot lattice) for (a) $P = 0$; (b) $P = 1$; and (c) $P = 2$.

Figure 3.

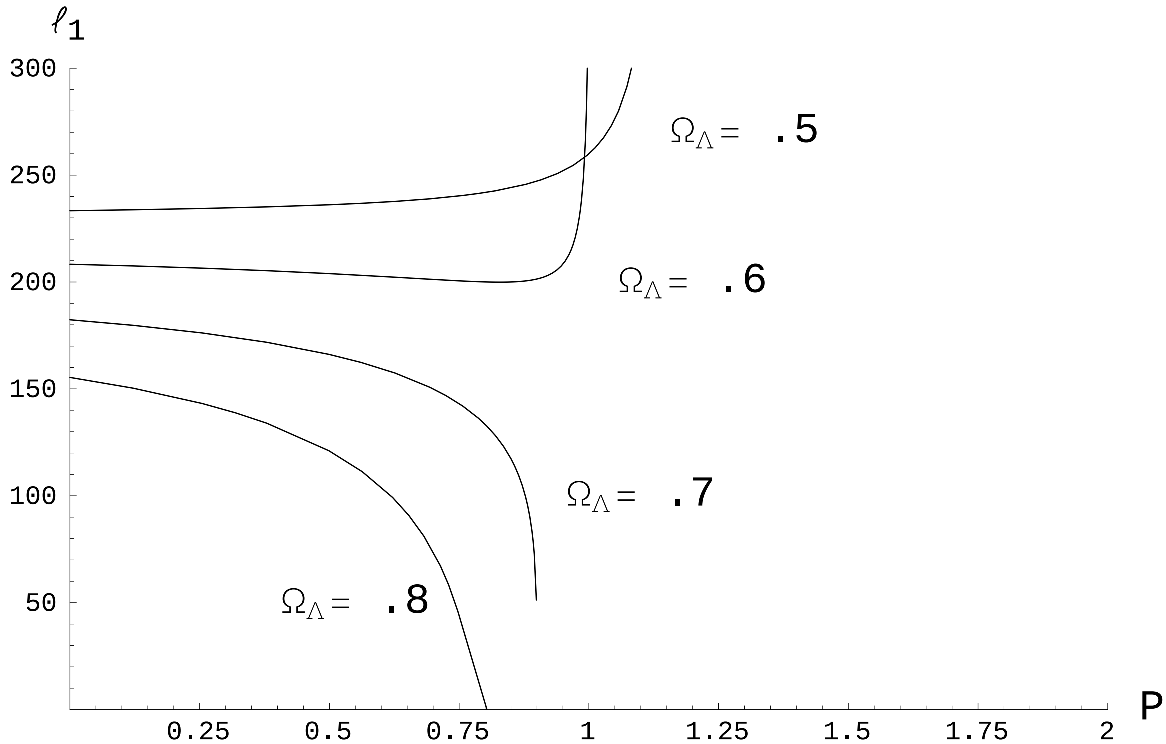
Enlarged view of $\Omega_M - \Omega_\Lambda$ plot to exhibit sensitivity to $0 \leq P \leq 0.2$.

Contours are (right to left) $P = 0, 0.05, 0.10, 0.15, 0.20$.

Figure 4.

Age of the universe in units $10^{10}y$ versus P.

This figure assumes $\Omega_M = 0.3$ and flatness $\Omega_C = 0$.



ℓ_1
300
250
200
150
100
50
$\Omega_\Lambda =$ .5
$\Omega_\Lambda =$ .6
$\Omega_\Lambda =$ .7
$\Omega_\Lambda =$ .8
0.25
0.5
0.75
1
1.25
1.5
1.75
2
P

Figure 1.(a)

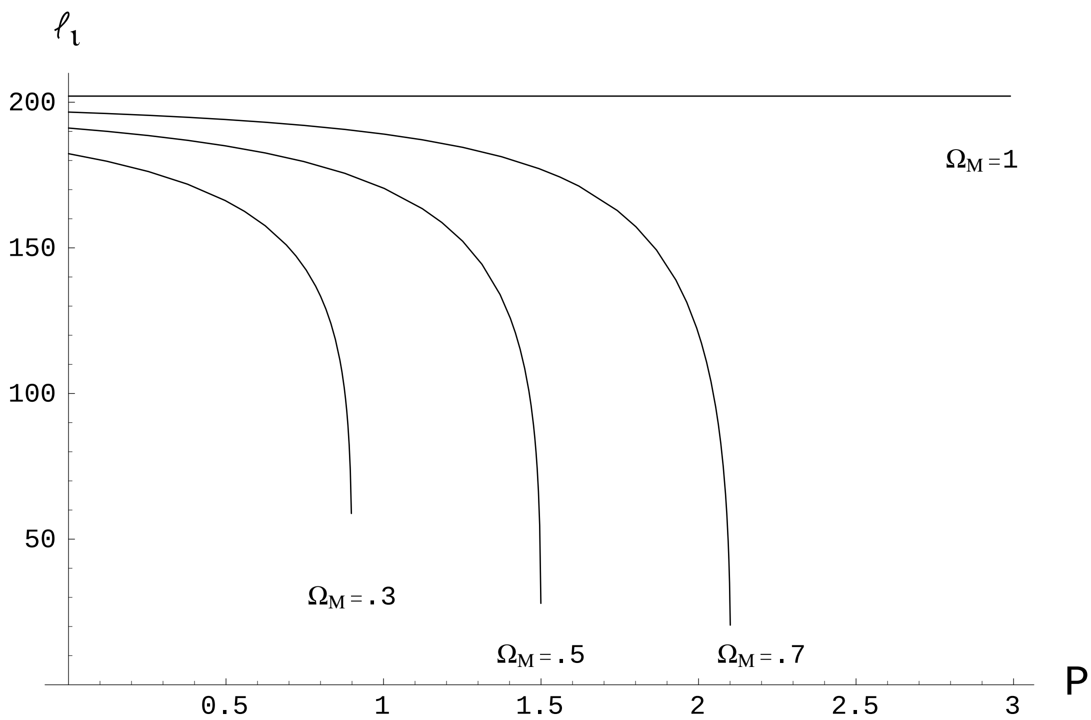
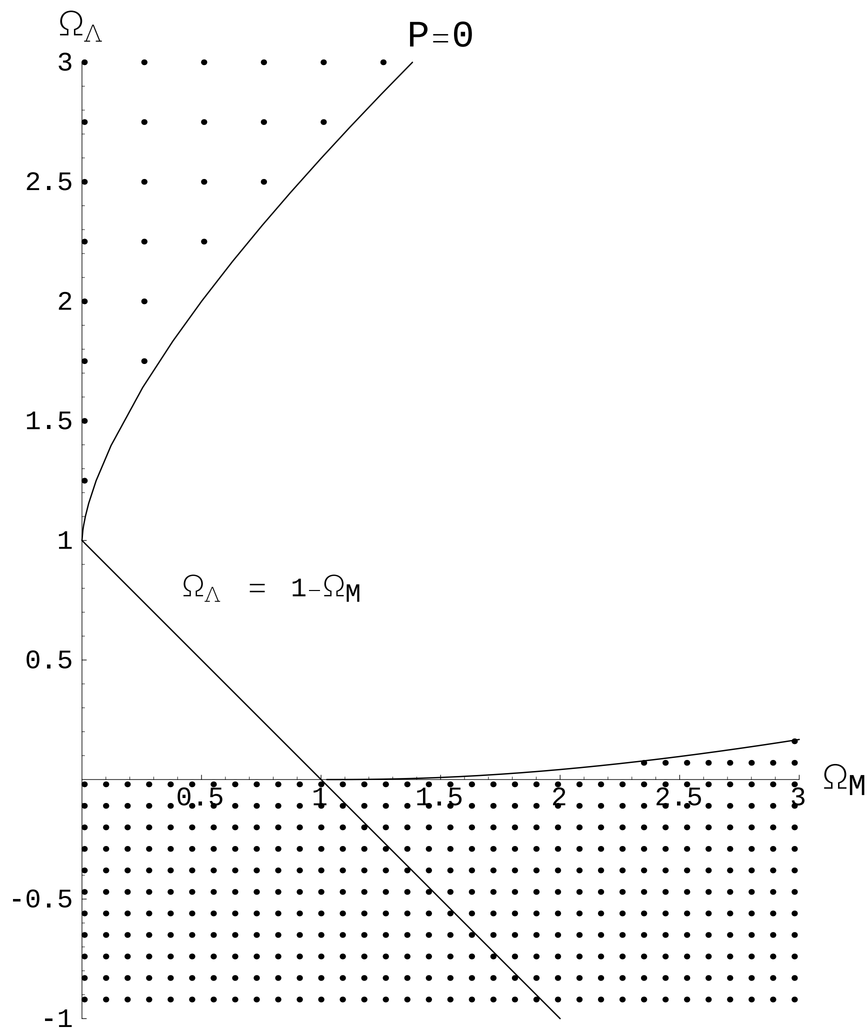

ℓ_1
200
150
100
50
$\Omega_M=1$
$\Omega_M=.3$
$\Omega_M=.5$
$\Omega_M=.7$
0.5
1
1.5
2
2.5
3
P

Figure 1. (b)



Ω_Λ
P=0
$\Omega_\Lambda = 1 - \Omega_M$
Ω_M
3
2.5
2
1.5
1
0.5
-0.5
-1
0.5
1
1.5
2
2.5
3

Figure 2. (a)

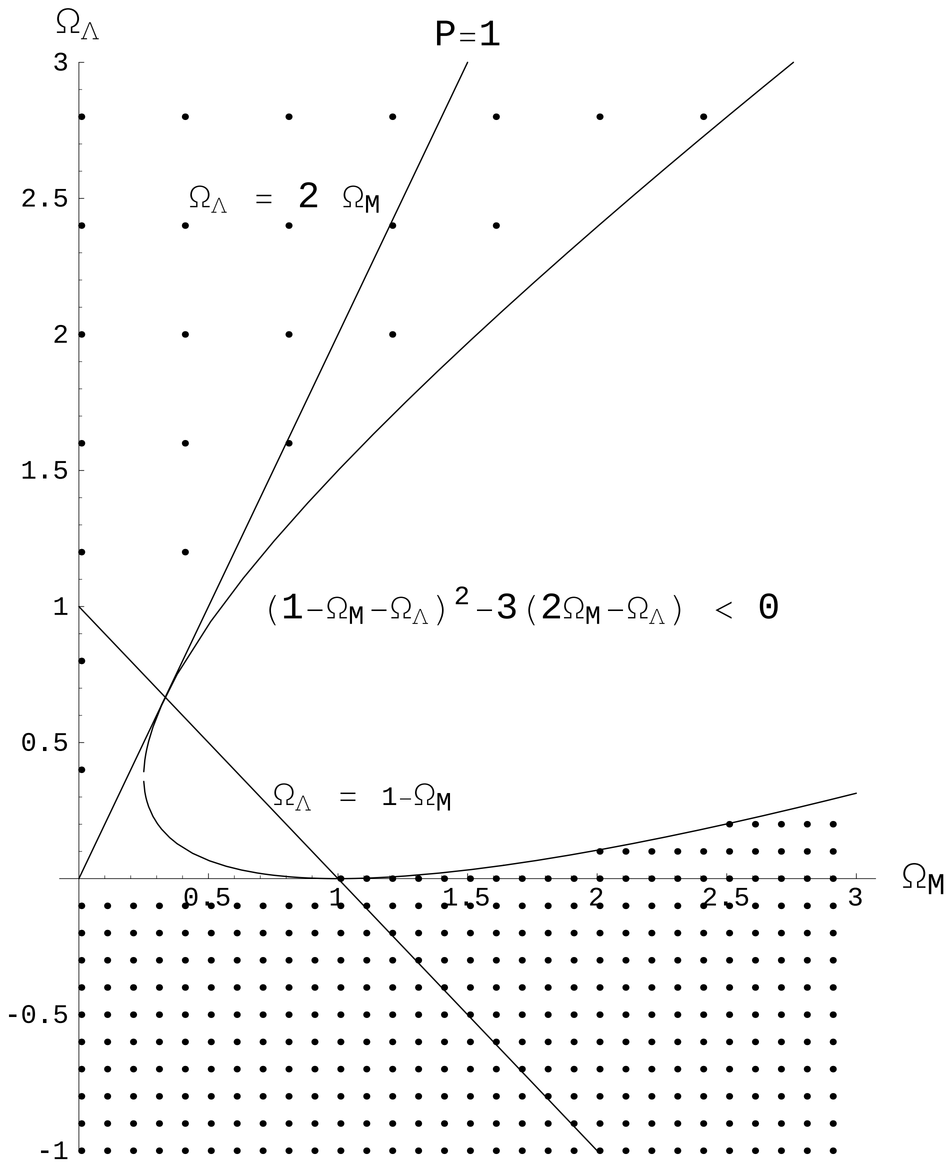
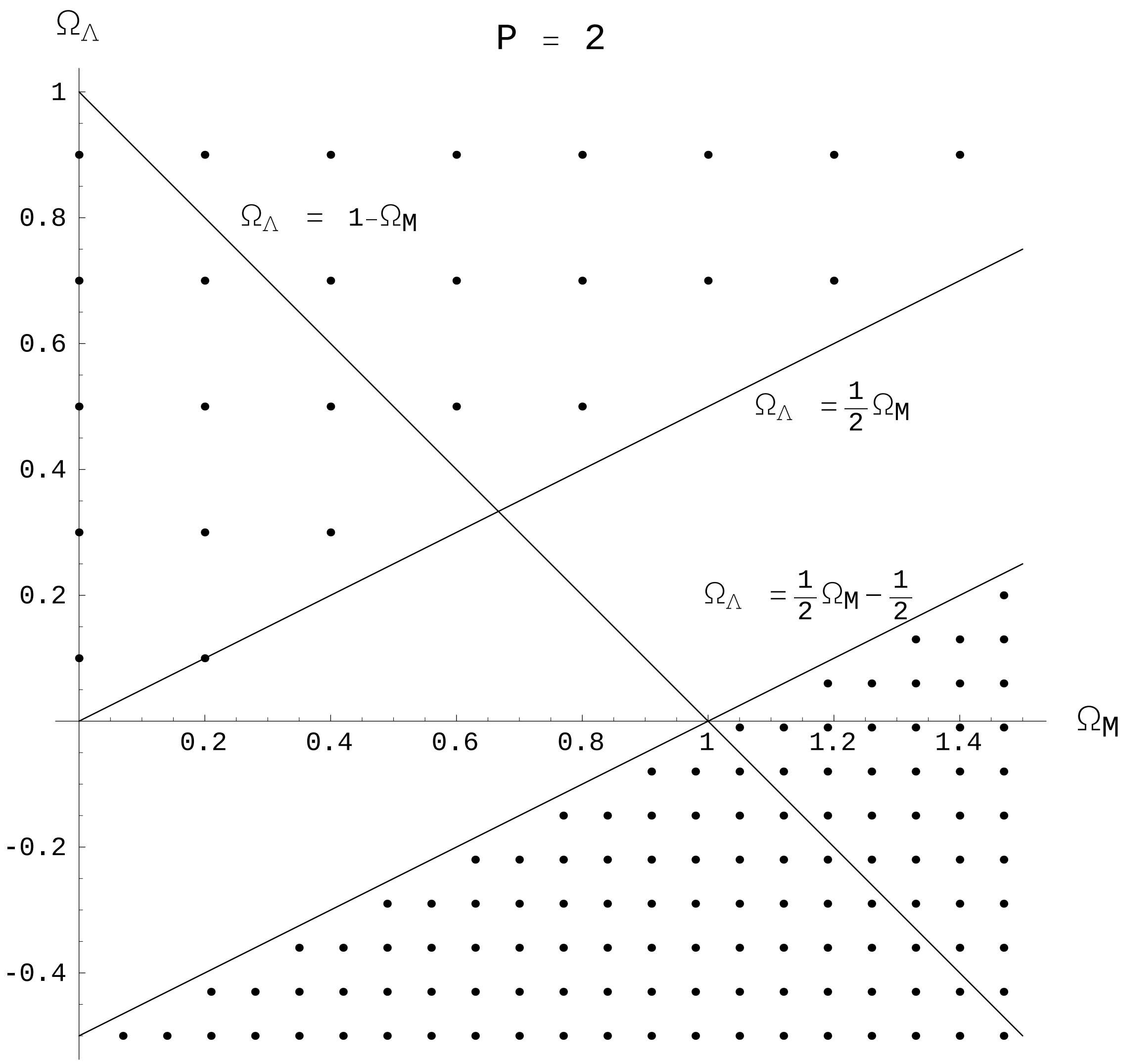

Ω_Λ
P=1
$\Omega_\Lambda = 2\,\Omega_M$
$(1-\Omega_M-\Omega_\Lambda)^2-3(2\Omega_M-\Omega_\Lambda) < 0$
$\Omega_\Lambda = 1-\Omega_M$
Ω_M
3
2.5
2
1.5
1
0.5
-0.5
-1
0.5
1
1.5
2
2.5
3

Figure 2. (b)



P = 2
Ω_Λ
Ω_M
$\Omega_\Lambda = 1 - \Omega_M$
$\Omega_\Lambda = \frac{1}{2}\Omega_M$
$\Omega_\Lambda = \frac{1}{2}\Omega_M - \frac{1}{2}$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 2. (c)

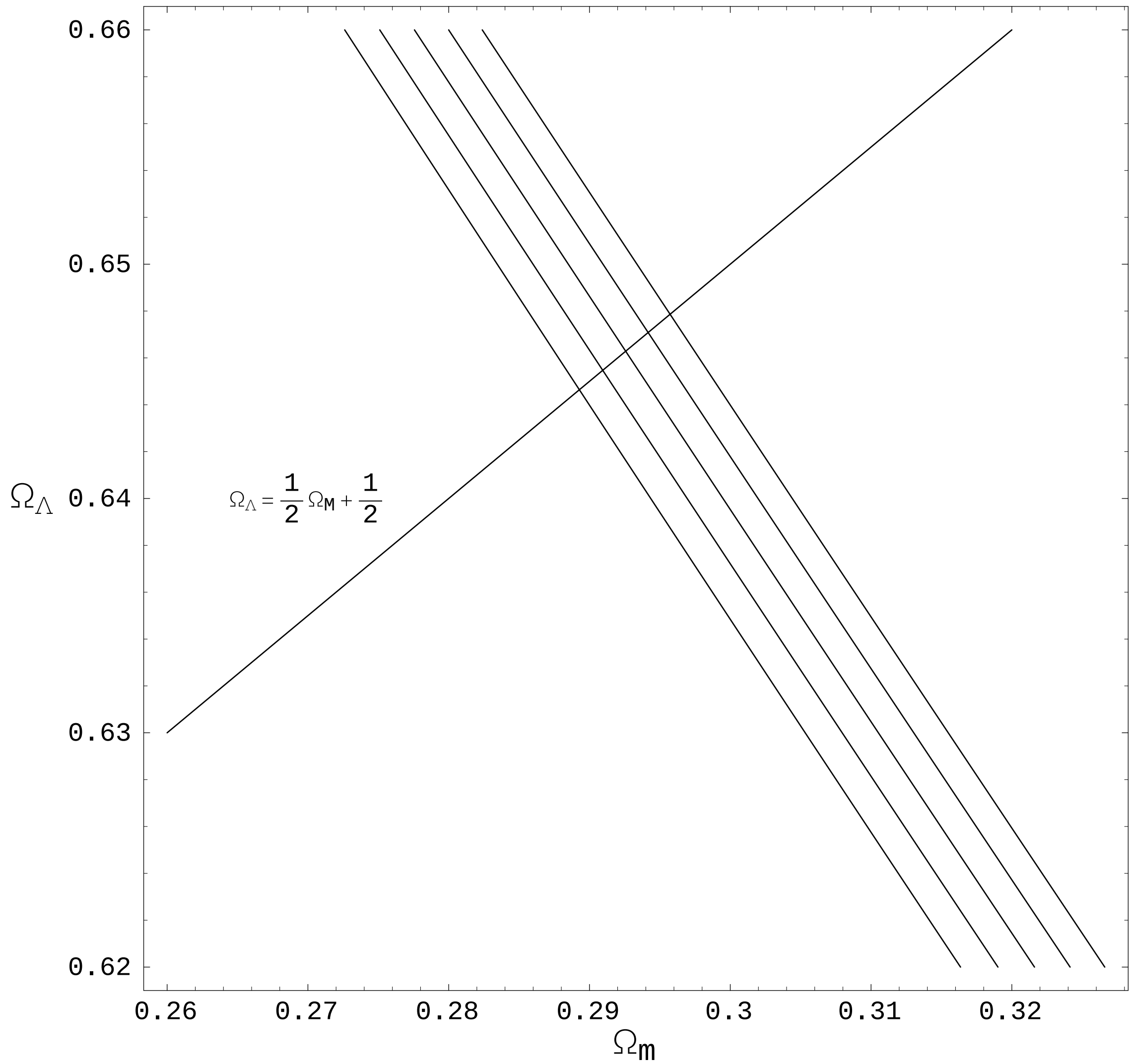

0.66
0.65
0.64
0.63
0.62
Ω_Λ
$\Omega_\Lambda = \frac{1}{2}\Omega_M + \frac{1}{2}$
0.26
0.27
0.28
0.29
0.3
0.31
0.32
Ω_m

Figure 3.

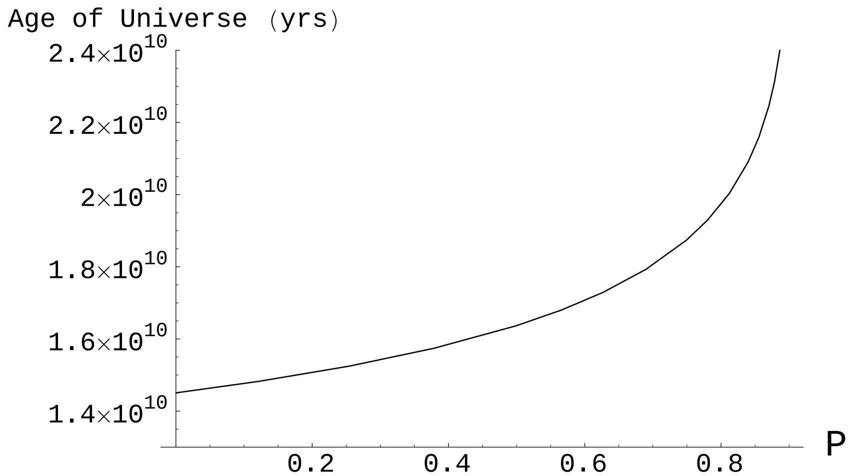

Age of Universe (yrs)
2.4×10^10
2.2×10^10
2×10^10
1.8×10^10
1.6×10^10
1.4×10^10
0.2
0.4
0.6
0.8
P

Figure 4.